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Journal of Computational Physics 209 (2005) 787–795

JOURNAL OF COMPUTATIONAL PHYSICS

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Capturing shock waves in inelastic granular gases

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Received 21 January 2005; received in revised form 30 March 2005; accepted 7 April 2005
Available online 23 May 2005

Abstract

Shock waves in granular gases generated by hitting an obstacle at rest are treated by means of a shock capturing scheme that approximates the Euler equations of granular gas dynamics with an equation of state (EOS), introduced by Goldshtein and Shapiro [J. Fluid Mech. 282 (1995) 75–114], that takes into account the inelastic collisions of granules. We include a sink term in the energy balance to account for dissipation of the granular motion by collisional inelasticity, proposed by Haff [J. Fluid Mech. 134 (1983) 401–430], and the gravity field added as source terms. We have computed the approximate solution to a one-dimensional granular gas falling on a plate under the acceleration of gravity until the close-packed limit.

MSC: 65M06; 76L05; 65Z05

Keywords: Granular gases; Inelastic collisions; Close-packed limit; Shock waves

1. Introduction

Much experimental and theoretical work has been performed to study the fluid properties of granular gases [1,3,6,7,10,11,14,18]. Several kinetic models have been introduced to explain the complicated physical behavior of granular media [15]. Continuum models, up to Navier–Stokes order, were derived from kinetic theory in [11]. Shock waves are one of the difficult features appearing in fluidized granular gases and easily

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[1] Research supported by DGICYT Project BFM2001-2814 and ONR N00014-03-1-0071.

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2005.04.004

observed in laboratory, since typical speeds of sound of some granular gases are measured in cm/s [19]. Hydrodynamical models are the most convenient and efficient ones to describe shock waves [6,8,9,11,13].

In this research work we are interested in simulating shock wave dynamics numerically using the Euler equations for compressible granular flow described by means of the granular equation of state (EOS) proposed to compute the pressure by Goldshtein and Shapiro [6], that includes both dense gas and inelastic effects. This granular EOS represents a particle-laden fluid through a volume fraction and its formulation is simple from the analytical and computational point of view. The granular gas described by this EOS can be considered, in a simple way, as the mixture of gas and particles, behaving as a gas for small volume fraction. In spite of particles and gas can not separate, this formulation appears to be computationally more advantageous than the two-phase approach, which would allow particle–gas separation. In the last case we would need two sets of equations and a mixture law. Further discussion of this issue is outside the scope of this research work.

We shall use an energy loss term, proportional to $T^{\frac{3}{2}}$, where T is the granular temperature [10], that takes into account the inelastic collisions of particles. We also consider the possible effect of the acceleration of gravity added as a source terms in both the momentum equation and the energy equation. The above hydrodynamic model was designed to describe the fluid-like properties of granular flows of a vibrated bed, and to be able to take into account the physical mechanism responsible for the transformation of the kinetic energy applied on the vibrating bed into granular temperature.

The main goal of this paper is twofold. We check that the model for inelastic granular gases has the necessary analytical properties to describe the shock wave phenomena by means of a shock capturing scheme. On the other hand, we propose a simple numerical scheme suitable for the present model, that uses all the wave structure information, behaves in a stable fashion, propagates discontinuities with correct speed and approximates the physically consistent solution under the presence of the gravity field and the energy loss by inelastic collisions.

We study the necessary thermodynamical properties of the granular EOS as the adiabatic exponent, the Grüneisen coefficient and the fundamental derivative, to ensure a unique solution of the Riemann problem for this model. We obtain analytical expressions of these variables showing that the nonlinear characteristic fields are genuinely nonlinear with positive nonlinearity [22].

The paper is organized as follows. In Section 2, we set up the model equations and analyze the thermodynamical variables associated to the EOS relevant for the propagation of acoustic waves. In Section 3, we describe the algorithm used and we analyze the reflected shock wave generated when a granular gas hits a solid wall under the acceleration of gravity. In Section 4, we draw our conclusions.

2. Euler equations for compressible granular flows

To simplify the discussion we restrict our analysis in this section to one spatial dimension. The one-dimensional Euler equations for inelastic granular flow can be written as:

$$\begin{aligned}
&\rho_t + (\rho u)_x = 0, \\
&(\rho u)_t + \left(P + \frac{(\rho u)^2}{\rho}\right)_x = \rho g, \\
&E_t + (u(E+P))_x = -\Theta + \rho g u,
\end{aligned}$$

where ρ is the granular gas density, u is the velocity, P is the pressure, Θ is the energy loss term and E is the total granular energy, $E = \frac{1}{2}\rho u^2 + \rho\epsilon$ being ϵ the specific internal energy per unit of volume. We shall use a granular equation of state (EOS), introduced by Goldshtein and Shapiro [6], to compute the pressure, that reads as follows: let σ be the diameter of particles of fixed mass and let e be their restitution coefficient ($0 \leqslant e \leqslant 1$). Let $v = \frac{\pi}{6}\rho\sigma^3$ be the *volume fraction*, with $v_{\max} = 0.65$ the maximum possible solids volume per unit volume of gas. Then, we have the following expression for the granular EOS:

$$P = T\rho A(\rho), \tag{1}$$

where $T = (\gamma - 1)\epsilon$ is the granular temperature, γ is the ratio of specific heats for the ideal gas case (in this paper, we use $\gamma = 5/3$) and $A(\rho) = 1 + 2(1 + e)G(v(\rho))$, where

$$G(v) = v\left[1 - \left(\frac{v}{v_{\max}}\right)^{\frac{4}{3}v_{\max}}\right]^{-1}.$$

The energy loss term Θ accounts for inelastic collisions. It is described by an extension of the so-called *Haff's cooling law* [10], in the form:

$$\Theta = \frac{12}{\sqrt{\pi}}(1 - e^2)\frac{\rho T^{\frac{3}{2}}}{\sigma}G(v). \tag{2}$$

For the elastic limit $e = 1$ this term has no effect.

We can associate to the granular EOS (1) a well-defined thermodynamic speed of sound, c_s, from the expression:

$$c_s^2 = (\gamma - 1)\epsilon\big(A(\rho) + \rho A'(\rho) + (\gamma - 1)A^2(\rho)\big), \tag{3}$$

where

$$A'(\rho) = \frac{\pi}{6}\sigma^3(1 + e)\left(1 + \left(\frac{4}{3}v_{\max} - 1\right)\left(\frac{v}{v_{\max}}\right)^{\frac{4}{3}v_{\max}}\right)\left[1 - \left(\frac{v}{v_{\max}}\right)^{\frac{4}{3}v_{\max}}\right]^{-2}$$

and the characteristic speeds: $u - c_s$, u, and $u + c_s$.

For $0 \leqslant v < v_{\max}$ it is easy to see that $c_s^2 > 0$ and it is a non negative strictly increasing function of v, such that $\lim_{v \to v_{\max}} c_s = +\infty$, for constant T. This shows that for *volume fractions* near $v_{\max}$ the granular gas becomes less compressible.

Since $c_s > 0$, the system is strictly hyperbolic and the waves are propagated with a uniquely defined finite speed.

The solution of the Riemann problem for a hyperbolic system is the fundamental ingredient for the design of a shock capturing numerical scheme. Thus, we need to know that the hyperbolic model for granular gas described above has the necessary analytical properties to ensure that there is a unique well-defined standard solution of the Riemann problem. For this purpose we analyze the thermodynamic magnitudes of the granular EOS relevant for the study of the wave propagation structure.

When heat conduction is neglected as in our case the properties of the shock waves and rarefaction waves are determined by the adiabatic exponent, γ_A, the Grüneisen coefficient, Γ and the fundamental derivative, $\mathcal{G}$ [17,22].

These thermodynamic variables are defined as, $\gamma_A := -\frac{V}{P}\frac{\partial P}{\partial V}\big|_S$, $\Gamma := V\frac{\partial P}{\partial \epsilon}\big|_V$ and $\mathcal{G} := \frac{1}{2}\frac{V^2}{\gamma_A P}\frac{\partial^2 P}{\partial V^2}\Big|_S$, where $V = \frac{1}{\rho}$.

We obtain, after straightforward calculations, the following expressions for the granular EOS of γ_A, Γ and $\mathcal{G}$, in terms of the volume fraction,

$$\gamma_{\mathrm{A}} = \gamma(1+2(1+e)G(v)) + \frac{2(1+e)G(v)}{1+2(1+e)G(v)}\left[\frac{\frac{4}{3}v_{\max}\left(\frac{v}{v_{\max}}\right)^{\frac{4}{3}v_{\max}}}{1-\left(\frac{v}{v_{\max}}\right)^{\frac{4}{3}v_{\max}}}\right], \tag{4}$$

$$\Gamma = (\gamma-1)(1+2(1+e)G(v)), \tag{5}$$

$$\mathscr{G} = \frac{1}{2}\left[1+\gamma_{\mathrm{A}}+\frac{v}{\gamma_{\mathrm{A}}}\left.\frac{\partial\gamma_{\mathrm{A}}}{\partial v}\right|_{\mathrm{S}}\right], \tag{6}$$

where

$$\left.\frac{\partial\gamma_{\mathrm{A}}}{\partial v}\right|_{\mathrm{S}} = 2(1+e)G'(v)\gamma + \frac{2(1+e)\left(\frac{4}{3}\right)^2(v_{\max})^3\left[\left(\frac{v}{v_{\max}}\right)^{\frac{4}{3}v_{\max}}+\frac{v}{v_{\max}}\right]}{[1+2(1+e)G(v)]\left(1-\left(\frac{v}{v_{\max}}\right)^{\frac{4}{3}v_{\max}}\right)^3} + \left[\frac{\frac{4}{3}v_{\max}\left(\frac{v}{v_{\max}}\right)^{\frac{4}{3}v_{\max}}}{1-\left(\frac{v}{v_{\max}}\right)^{\frac{4}{3}v_{\max}}}\right]\frac{2(1+e)G'(v)}{[1+2(1+e)G(v)]^2}.$$

In Fig. 1, we display the adiabatic exponent and the fundamental derivative as functions of the *volume fraction* in the interval [0,0.55], for two different values of the restitution coefficient $e = 0.9, 0.2$, using 100 points. The profiles of these thermodynamic quantities are non negative strictly increasing functions of the *volume fraction*, tending to infinity when v tends to $v_{\max}$ and their minimum values are the corresponding ones for the ideal gas case.

Since the fundamental derivative is strictly positive and the granular EOS satisfies the Menikoff–Plohr "strong condition" [17, p. 95] $\Gamma = PV/\epsilon$, we conclude that the Riemann problem has a unique standard solution and the nonlinear characteristic fields are genuinely nonlinear with a positive nonlinearity [22]. Furthermore, since $\mathscr{G} > 1$, the isentropes in the P–ρ plane are convex, and, since $\Gamma > 0$ the isentropes do not cross each other in the P–V plane. Thus, the "exact" solution of the Riemann problem can be approximated in a simpler way. This means that the solution of the Riemann problem for the granular EOS has an

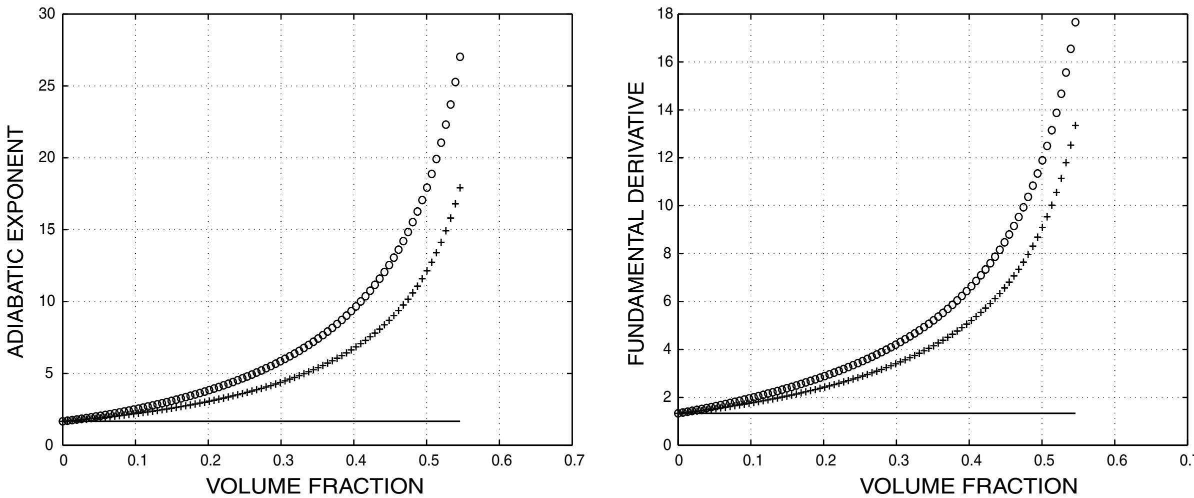

Fig. 1. Left: volume fraction vs. adiabatic exponent. Right: volume fraction vs. fundamental derivative, $e = 0.9$: '○'; $e = 0.2$: '+'.

analogous structure to the one for ideal gas dynamics but the evolution of shock waves and their interactions are more complex since the three fundamental thermodynamic quantities studied above are nonconstant functions of the volume fraction.

We can recover the ideal gas EOS by putting $\sigma = 0$. Hence, the thermodynamic quantities are constant with respect to the density ρ and their values are $\gamma_A = \gamma$, $\Gamma = \gamma - 1$ and $\mathcal{G} = \frac{1}{2}(1+\gamma)$. When volume fraction is very small the granular gas resembles an ideal gas, since γ_A, Γ and $\mathcal{G}$, are close to the minimum values γ, $\gamma - 1$ and $\frac{1}{2}(1+\gamma)$, respectively.

In addition to the qualitative properties of the granular EOS discussed above, we can assert that the main difference between this model for inelastic granular gas and the Euler equations for ideal gas is the presence of the energy loss term. The kinetic energy loss by inelastic collisions is transformed in granular temperature. Thus, total energy is not conserved and this feature makes the physics of granular media more complex [10,15].

3. Numerical experiment

We express in short an initial value problem for the one-dimensional hyperbolic system of conservation laws describing granular flows as

$$\mathbf{u}_t + (\mathbf{f}(\mathbf{u}))_x = \mathbf{S}(\mathbf{u}), \tag{7}$$

together with the initial data

$$\mathbf{u}(x,0) := \mathbf{u}_0(x), \tag{8}$$

where $\mathbf{f}(\mathbf{u})$ is the flux vector and $\mathbf{S}(\mathbf{u})$ is the vector of source terms.

We consider the computational grid: $x_j = jh$ (h is the spatial step) $t_n = n\Delta t$, the time discretization, (Δt is the time step), $I_j = [x_{j-\frac{1}{2}}, x_j + \frac{1}{2}]$ is the spatial cell, where $x_{j+\frac{1}{2}} = x_j + \frac{h}{2}$ is the cell interface and $C_j^n = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t_n, t_{n+1}]$ is the computational cell. Let $\mathbf{u}_j^n$ be an approximation of the mean value in I_j, $\frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{u}(x,t_n)\,\mathrm{d}x$, of the exact solution $\mathbf{u}(x,t_n)$ of the initial value problems (7) and (8), obtained from a finite volume scheme in conservation form:

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{h}(\tilde{\mathbf{f}}_{j+\frac{1}{2}} - \tilde{\mathbf{f}}_{j-\frac{1}{2}}) + \Delta t \mathbf{S}(\mathbf{u}_j^n), \tag{9}$$

where the numerical flux, $\tilde{\mathbf{f}}$, is a function of $k + l$ variables

$$\tilde{\mathbf{f}}_{j+\frac{1}{2}} = \tilde{\mathbf{f}}(\mathbf{u}_{j-k+1}^n, \ldots, \mathbf{u}_{j+l}^n), \tag{10}$$

which is consistent with the flux of Eq. (7), $\tilde{\mathbf{f}}(\mathbf{u},\ldots,\mathbf{u}) = \mathbf{f}(\mathbf{u})$.

From the classic theorem of Lax and Wendroff we know that the limit solution of a consistent scheme in conservation form is a weak solution of the hyperbolic PDE system. Thus, a consistent scheme in conservation form is the main ingredient to design shock capturing schemes, since these schemes propagate discontinuities at the correct speed. In addition, we require a numerical scheme that provides the appropriate viscosity to be stable and to develop the physically consistent features of the shock wave phenomena. In order to construct an explicit scheme in conservation form we need a flux formula that approximates the numerical flux $\tilde{\mathbf{f}}$ at each cell interface.

In this paper, we use Marquina Flux Formula, (MFF) [4], for the design of our numerical scheme. MFF uses the information related to the wave structure through the spectral decomposition of the Jacobians of the flux computed at both neighboring cells. This flux formula computes the numerical flux by performing a characteristic field decomposition at $\mathbf{u}_l$ and $\mathbf{u}_r$, using Godunov's method for non-transonic wavefields and using local Lax–Friedrichs method for transonic ones prescribing the local viscosity as the maximum of the absolute values of the local characteristic speeds at neighboring cells (see [4] for details).

The most advantageous features of the MFF to resolve successfully the computations of strong shocks in granular flow are that it behaves robustly for low densities [4] and that can be applied to nonhomogeneous fluxes [5,21].

Higher order of accuracy is obtained by applying a reconstruction procedure on local variables or local fluxes extrapolating them to the left and right states of the cell interface following the so-called Shu–Osher "flux formulation", [20]. In this paper, we have used the PHM method [16] and we integrate in time using the third-order accurate Shu–Osher TVD Runge–Kutta time-stepping procedure [20]. The resulting scheme is stable under a Courant–Friedrichs–Lewy (CFL) restriction of the form $\lambda = \frac{\Delta t}{h} \leqslant \lambda_0$, where λ_0 is proportional to $\frac{1}{\max_{p,u}|\lambda_p(u)|}$ as usual.

The chosen RK procedure remains stable under the presence of the source terms, imposing no reduction of the time step. The source terms, including the gravitational field, are not stiff since they vary smoothly using the stepsize obtained from the CFL restriction. On the other hand, the gravitational term increases the absolute value of the velocity and, hence the CFL restriction of the hyperbolic part of the numerical scheme enforces a reduction of the time stepsize in an automatic way, ensuring stability.

We use "ghost cells" to implement boundary conditions. The values of conserved variables on these cells are computed by linear extrapolation of the corresponding values in the domain for the outflow/inflow boundary conditions. Reflective boundary conditions are computed using the same procedure although changing the sign of the momentum in the direction normal to the boundary.

In our calculations we fix the value of the diameter of the particles to be $\sigma = 0.1$. The role of σ in the continuous model is just a scale factor that relates the *volume fraction* and the density appearing in the Euler equations.

3.1. Gravity acceleration of granular gas hitting a solid wall until close-packed limit

We consider a one-dimensional domain of length 10 cm, [0,10], filled with a granular gas with a constant *volume fraction* $v = 0.018$, restitution coefficient $e = 0.97$ and constant speed of sound of 9 cm/s. We consider a solid wall at the right end (reflective boundary conditions are applied), and the action of the gravity field oriented from left to right. The initial velocity is constant through the domain and is taken to be 18 cm/s. The initial data were derived from the ones appearing in [19]. A shock wave is formed immediately at the solid wall and propagates to the left. The granular gas starts to cluster at the wall until reaches the close-packed limit. The initial data computed from the above quantities are $(\rho, v, P) = (34.37, 18, 1589.26)$. We use $g = 980\ \mathrm{cm/s^2}$.

We have performed our computation using 1000 grid points, until time 0.23 with a Courant–Friedrichs–Lewy (CFL) factor of 0.5 using MFF scheme with the third-order accurate PHM reconstruction. In Fig. 2 we display the *volume fraction*, the granular temperature, the pressure and the Mach number at time 0.23. We observe the rarefaction wave generated by the energy dissipation term where granular temperature becomes close to zero at the wall reaching a *volume fraction* of 0.649472.

The post-shock oscillations observed in the pressure profile are due to the high order accurate procedure used and the increase of the Mach number of the incoming granular gas.

In Table 1, we display the first-order approximations of the shock wave location for this experiment at times 0.14, 0.16 and 0.18 under mesh refinement. The observed convergence shows numerical evidence that the presence of source terms does not affect the correct speed and position of the shock wave.

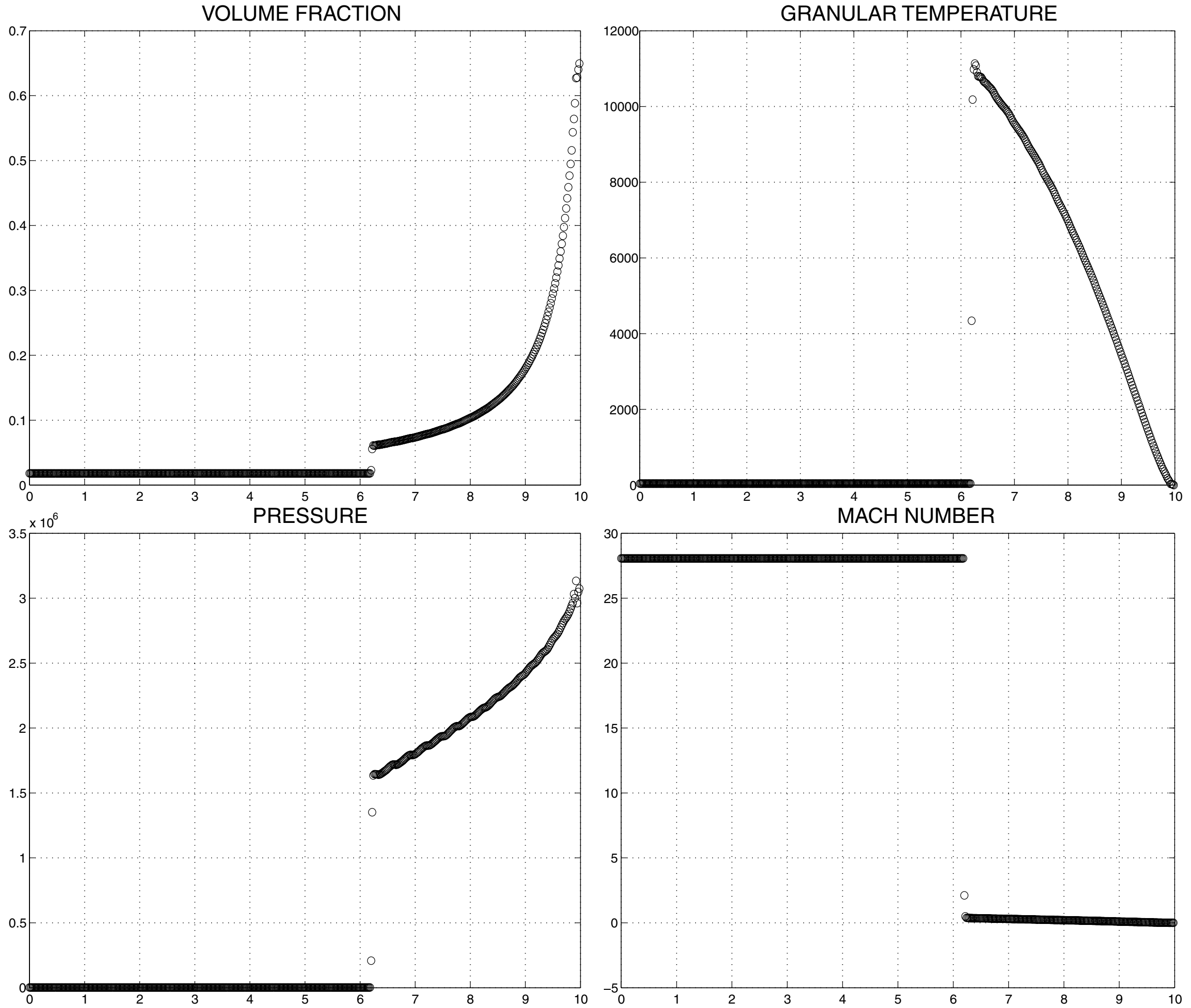

Fig. 2. Inelastic granular gas falling on a plate (right end) under the action of gravity at time 0.23 with $e = 0.97$ and MFF–PHM scheme. Top left: *volume fraction*; top right: granular temperature; bottom left: pressure, bottom right: Mach number.

Our method behaves stable and accurate for nonzero velocity initial data using central values to compute the source terms (gravity field).

The accuracy and stability of our method for static initial data in comparison with standard well-balanced schemes [2,12] applied to the present model is a work in progress.

Table 1
Abscissas of shock wave location at different times

Time	Position			
	$N = 200$	$N = 400$	$N = 800$	$N = 1600$
0.14	7.8250	7.8375	7.8312	7.8343
0.16	7.4250	7.4375	7.4312	7.4406
0.18	7.0250	7.0375	7.0562	7.0593

4. Conclusions

In this research work, we have shown that Euler equations together with an energy loss term and an equation of state representing the fluidized granular gas until the close-packed limit, have the necessary analytical properties to describe shock wave phenomena by means of a shock capturing scheme. We have used a standard numerical scheme, based on an approximate Riemann solver to compute the approximate solution to a one-dimensional granular gas falling on a plate under the acceleration of gravity until close-packed limit. Further numerical experiments on shock wave propagation in inelastic granular gases are the object of future work.

Acknowledgments

This research was carry out while the authors were visiting UCLA Department of Mathematics and they are very grateful to the people of the Computational and Applied Mathematics research group for their hospitality.

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